Journal of Statistical Physics, Vol. 15, No. 2, 1976

Ensemble Mechanics for the Random-Forced Navier–Stokes Flow

Iwao Hosokawa[1]

Received July 17, 1975; revised January 5, 1976

It is shown that the so-called statistical theory of turbulence, which can be exactly incorporated by the Hopf functional equation, is imperfect in that it fails to ensure the irreversible approach to a unique ultimate steady state of turbulence (for a steady boundary condition) expected from observation; and that this imperfection is removed if a stochastic random-force term is added into the Navier–Stokes equation. The ensemble mechanics for the random-forced Navier–Stokes flow is formulated by taking into account the natural random force, which has usually been neglected in the Navier–Stokes equation.

KEY WORDS: Ensemble mechanics; turbulence; Hopf equation; characteristic functional; random-forced Navier–Stokes flow; Markovian stochastic process.

1. INTRODUCTION

The significance of a random-forced Navier–Stokes flow has occasionally been considered from the physical point of view,[1–4] besides the artificial introduction of a random force into the Navier–Stokes equation in some studies of turbulence.[5–8] To physicists, it is obvious that the Navier–Stokes equation is never perfect but is missing some fluctuation term, which should include what may be called the thermal agitation caused by molecular motion. Since this term is exceedingly small at normal temperatures, it has been conventionally thought that it is unnecessary to take it into consideration in order to describe typical laminar motions of a fluid. The past success of the Navier–Stokes equation in deriving many practically useful laminar-flow solutions has been so great that many people still believe every behavior of a fluid, *including turbulence*, to be perfectly governed by the Navier–Stokes

[1] Department of Mechanical Engineering, Iwate University, Morioka, Japan.

© 1976 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording, or otherwise, without written permission of the publisher.

equation. On this belief, the statistical theory of turbulence[9] originated and was developed since Reynolds, keeping the concept of an ensemble of exact solutions for a vague set of possible initial conditions and appealing to the ergodic hypothesis, which assumes the equality of the ensemble average and the time average of the flow behavior. This theory was finally formulated by Hopf[10] in such a simple way that the whole ensemble of flows is governed by one single *closed* functional-differential equation for the characteristic functional of the velocity-field distribution. Thus, it may be said that it is often believed that the solution of the Hopf equation would expose all the details of a turbulent flow.

However, such a general belief is not free from some crucial difficulties. Does the Hopf equation guarantee the uniqueness of an ultimate steady state of the ensemble? If not, every state depends on an initial condition, and then what initial condition is best suited to bring forth a state in accord with observation? Is the ensemble mechanics really ergodic? Unfortunately, the present investigation gives these questions negative answers, as will be described in Section 2.

We are then in a position to reconsider the problem of whether the Navier–Stokes equation is usable for a turbulent (or unstable) flow case. Does the small missing term play a greater role than expected? Betchov[4] first pointed out the possibility that this term can act as a *continuous* trigger to push an unstable flow into turbulence. In order to see the stochastic effect of this term in a general way, we can most conveniently use the simple functional formulation of turbulence mechanics with the action of a random force, which was provided in earlier studies of turbulence[6–8] as an extended version of the Hopf formulation. In Section 3, this formulation is briefly sketched and the general effect of the random force is discussed. Here, it will be clear that the stochastic missing term has a decisive role in giving the ensemble mechanics irreversibility as well as ergodicity, however small it might be. It is, indeed, this term that makes our mechanics not deterministic but probabilistic, so that it reveals an ergodic Markovian stochastic process of fluid variables. A proper explicit form of the natural random force is shown and discussed in Section 4, which completes the ensemble mechanics in this presentation.

In the course of the treatment, the cylinder functional approach to the characteristic functional is occasionally retained, in order to avoid a difficulty of treating an infinite-dimensional probability density. All the probabilities of the velocity field we treat are finite-dimensional within this approach. On this basis, the nature of our Markovian stochastic process will be clarified; e.g., a many-dimensional Fokker–Planck equation for the probability density is constructed, which implies a generalized random-walk motion of the (approximate) velocity field. However, since a cylinder characteristic functional itself converges (in some sense described in Section 2) to the noncylinder limit, then a symbolic treatment of the corresponding infinite-dimensional probability is acceptable only for the conceptual convenience in showing that it is in parallel with the exact treatment of a noncylinder characteristic functional. For this reason, such a treatment is given with some care.

2. LIMITATIONS OF THE HOPF FORMALISM

Let χ be the vectorial Navier–Stokes operator such that $\partial u/\partial t = \chi(u)$ is the incompressible Navier–Stokes equation with the pressure term re-expressed by u, where $u(x)$ is the velocity vector field in physical space and t is the time variable. Then, the Hopf equation may be expressed as

$$\frac{\partial \psi}{\partial t} = i \int y_\alpha(x) \chi_\alpha \left[\frac{\delta}{i\, \delta y(x)} \right] \psi \, dx \tag{1}$$

where $\psi(y, t)$ is the characteristic functional for the stochastic field $u(x)$ and $\delta/\delta y(x)$ denotes functional differentiation with respect to a real vector function $y(x)$. Hereafter, Greek subscripts indicate components of a vector or tensor and follow the summation convention. The associative conditions for ψ to be a characteristic functional are

$$\psi(0, t) = 1 \tag{2}$$

$$\psi^*(y, t) = \psi(-y, t) \tag{3}$$

the asterisk denoting the complex conjugate, and

$$|\psi(y, t)| \leqslant 1 \tag{4}$$

Furthermore, ψ obeys the incompressibility condition:

$$\operatorname{div} \frac{\delta \psi}{i\, \delta y(x)} = 0 \tag{5}$$

Next, we write down the inverse Fourier transform of the Hopf equation in the cylinder functional approach. We consider $y(x)$ in the form

$$y_\alpha{}^M(x) = \sum_{i=1}^{M} a_{\alpha i} s_i(x) \tag{6}$$

with M finite, where $\{s_i(x)\}$ is an orthonormal function set in physical space; and we define a cylinder functional as

$$\psi^M(y, t) \equiv \psi(y^M, t) \tag{7}$$

which is a functional of y [through $a_{\alpha i} = \int s_i(x) y_\alpha(x)\, dx$] but may also be considered as a $3M$-dimensional function of $\{a_{\alpha i}\}$. Then, the inverse transformation

$$p^M = \int \psi^M \exp\left(-\sum_{i=1}^{M} i a_{\alpha i} b_{\alpha i}\right) \delta y^M \tag{8}$$

can be performed straightforwardly when δy^M is defined as

$$\delta y^M = \prod_{\alpha} \prod_{i=1}^{M} [da_{\alpha i}/(2\pi)^{1/2}] \tag{9}$$

p^M has the meaning of the probability density assigned by its characteristic function ψ^M. On this basis, from the Hopf equation (1), we have

$$\frac{\partial p^M}{\partial t} = -\sum_{i=1}^{M} \frac{\partial}{\partial b_{\alpha i}} (\chi_{\alpha i}^M p^M) \tag{10}$$

keeping in mind the relation

$$\frac{\delta}{\delta y_\alpha(x)} = \sum_{i=1}^{M} s_i(x) \frac{\partial}{\partial a_{\alpha i}} \tag{11}$$

and denoting

$$\chi_{\alpha i}^M = \int s_i(x) \chi_\alpha(u^M)\, dx \tag{12}$$

Here u^M is expressed in terms of $\{b_{\alpha i}\}$ using the same approach as in (6). This "probability" equation is sometimes simpler, since it does not include any imaginary factor. Now, we find the characteristic base curves of the partial differential equation (10) by solving a set of equations

$$\frac{db_{\alpha i}}{dt} = \chi_{\alpha i}^M \tag{13}$$

which is equivalent to $\partial u/\partial t = \chi(u)$ as $M \to \infty$. This means that the characteristic base curves in the function (u^M)–time space are nothing but general solutions of the approximate Navier–Stokes equation. Thus, the function p^M and then ψ^M always depend on particular initial conditions through these characteristic base curves, and then it is difficult, in general, to expect that all the ψ^M merge to the same one, irrespective of initial conditions. The situation is not unchanged as $M \to \infty$, provided that $\{\psi^M(y)\}$ forms a Cauchy sequence.

When all y belong to the Banach space L_1 in which $\int |y(x)|\, dx < \infty$, and when ψ is analytic in this space so that the functional derivative of ψ is bounded, the fact is proved, because we have the relation that, for $M > N$,

$$\psi^M(y) - \psi^N(y) = \int \frac{\delta\psi[y^N + \theta(y^M - y^N)]}{\delta y(x)} [y^M(x) - y^N(x)]\, dx \tag{14}$$

with $0 < \theta < 1$ (by virtue of the functional version of the mean-value theorem), which leads to

$$|\psi^M - \psi^N| < C\|y^M - y^N\| \tag{15}$$

for all $y \in L_1$, where C is a fixed value larger than $\sup|\delta\psi/\delta y(x)|$ and

$$\|y^M - y^N\| = \int |y^M(x) - y^N(x)|\, dx \tag{16}$$

Obviously, $\{\psi^M\}$ forms a Cauchy sequence with respect to the maximum norm for functionals; in other words, ψ^M is convergent as $M \to \infty$, preserving the same analytical quality as ψ^M. In contrast, $\{p^M(u)\}$ is hardly convergent in the same sense. Nonetheless, the use of p without M will be allowed in our notation, e.g., in place of (10) we may write

$$\frac{\partial p}{\partial t} = -\int \frac{\delta}{\delta u_\alpha(x)}\,[\chi_\alpha(u)p]\, dx \tag{17}$$

which is in parallel with (1). In this case, p itself has no meaning, except for the convergence of $\{\psi^M\}$.

Thus, every ψ generally depends on an initial condition and does not guarantee the unique existence of an ultimate steady state. This result is actually due to the fact that (10) is a first-order differential equation.

The only exception is for a subcritical Reynolds number, when we expect a laminar steady state to appear ultimately. This case is realized by the special arrangement of all the characteristic base curves such that they gather asymptotically with increasing time into the single line with $u^M = u_\infty{}^M(x)$, which expresses a unique, stable, laminar steady state as $M \to \infty$. The solution for the symbolic p in this case may be expressed as the delta functional[11] $\delta[u(x) - u_\infty(x)]$, which should correspond exactly to $\psi_\infty(y) = \exp[i\int y(x)u_\infty(x)\, dx]$. Note that if we have more than one stable steady state, $u_{\infty 1}$, $u_{\infty 2}$, etc., then p and ψ should depend on initial conditions again since any (normalized) linear combination of $\delta[u(x) - u_{\infty 1}(x)]$, $\delta[u(x) - u_{\infty 2}(x)]$, etc., can be a solution for p. Such a case really happens for the supercritical, space-limited, main-flow-interacting Burgers model flow; in this case, it was argued that the ensemble mechanics governed by the Hopf equation is not ergodic.[12]

In this connection, it may also be instructive to mention Tatsumi and Ikeda's proof[13] that the (information) entropy of an ensemble of *inviscid*, isotropic, incompressible Navier–Stokes flows should be invariant to time, although this fact is not surprising, since the Hopf equation for the inviscid flow case is time-reversible.[14] A brief sketch of this is given in an alternative way in the appendix. This means, of course, that an equilibrium state, such as given by Hopf's energy-equipartitioned solution,[10,15] can never be approached from any other state with a different entropy. (They derived further that the entropy always decreases with time without limitation if viscosity exists. This will be amended by taking account of the effect of thermal agitation. See the appendix.)

There is one way of removing such limitations of the Hopf formalism for turbulence. It is to modify the Navier–Stokes equation in a different form, e.g., to include some independent stochastic term f as in

$$\frac{\partial u}{\partial t} = \chi(u) + f \tag{18}$$

To consider an independent stochastic term in the governing dynamics for each realization leads, necessarily, to a functional formalism substantially different than the Hopf equation; examples are found in the work of Novikov,[6] Edwards,[7] and Hosokawa.[8] In these cases, the random force effect obviously stops time reversibility even for the inviscid flow case, as is seen in the next section. The functional formalism reveals a Markovian stochastic process played in the function space! The model Navier–Stokes dynamics in the direct interaction approximation by Kraichnan[16] seems to have a similar nature. In fact, starting from a Markovian stochastic process very close to what was introduced as representing the basic dynamics of Kraichnan's random coupling model, Frisch *et al.*[17] succeeded in constructing not exactly Kraichnan's, but a simpler random coupling model.

3. ENSEMBLE MECHANICS WITH RANDOM FORCE ACTION

According to previous work,[8] the basic equation for the characteristic functional $\psi(y, t)$ incorporating the effect of a general random force acting in the form of (18) is written as follows:

$$\frac{\partial \psi}{\partial t} = i \int y_\alpha(x) \left\{ \chi_\alpha \left[\frac{\delta}{i\, \delta y(x)} \right] + \mathscr{E}_\alpha(x, t) \right\} \psi \, dx + \Gamma(y, t)\psi \tag{19}$$

where $\mathscr{E}(x, t)$ is the ensemble average of the vectorial random force field $f(x, t)$ and

$$\Gamma(y, t) = \frac{i^2}{2!} \iint y_\alpha(x) y_\beta(x') F_{\alpha\beta}(x, x', t) \, dx \, dx' + \frac{i^3}{3!} \iiint y_\alpha(x) y_\beta(x') y_\gamma(x'') G_{\alpha\beta\gamma}(x, x', t) \, dx \, dx' \, dx'' + \cdots \tag{20}$$

with $F_{\alpha\beta}$, $G_{\alpha\beta\gamma}$,... denoting, respectively, the second-order, third-order,..., cumulants of correlation of the random force field at time t. The only assumption made on the random force field is that random forces at different times are stochastically independent; for example, we have

$$\langle f_\alpha(x, t) f_\beta(x', t') \rangle = \delta(t - t') F_{\alpha\beta}(x, x', t) \tag{21}$$

where the brackets denote the ensemble average. [$\mathscr{E}_\alpha$, $F_{\alpha\beta}$, $G_{\alpha\beta\gamma}$, etc., may depend on $u(x)$.]

Novikov[6] gave the same functional equation for the special case in which $\mathscr{E}(x, t) = 0$ and all the cumulants vanish except for $F_{\alpha\beta}(x, x', t) \equiv F_{\alpha\beta}(x - x')$, i.e., the case with a homogeneous stationary Gaussian random force. The Fourier-transformed expression of this same case was considered independently by Edwards[7] as a starting point of his approach to isotropic turbulence. He called it the generalized Liouville equation (averaged over the fluctuating force). It may be written symbolically in our physical space representation of velocity, as

$$\frac{\partial p}{\partial t} = -\int \frac{\delta}{\delta u_\alpha(x)} [\chi_\alpha(u) p]\, dx + \frac{1}{2} \int\int \frac{\delta^2}{\delta u_\alpha(x)\, \delta u_\beta(x)} [F_{\alpha\beta}(x - x') p]\, dx\, dx' \tag{22}$$

When we have no special reason for considering a more complicated random force, we take (22) as the basic equation.

It is important to notice that (22) is no longer a first-order differential equation, but second-order and parabolic; all curves given by (13) become meaningless. In other words, this is a Fokker–Planck or Kolmogorov equation, which describes a simple Markovian stochastic process played by a variable $u(x)$. Since F should be positive definite,[8] as is also known from (21), (22) is never time-reversible, even if χ is. On this point, Betchov's[4] conjecture that the effect of the random force f in (18) is time-reversible is completely wrong. Moreover, it is guaranteed[18] that a many-dimensional Fokker–Planck equation of the type (22) has a unique steady-state solution $p_\infty{}^M$ and that all states approach $p_\infty{}^M$ irreversibly. Hence, the corresponding solution $\psi_\infty{}^M$ exists and the noncylinder limit ψ_∞ of $\{\psi_\infty{}^M\}$ represents the exact final state of an ensemble of flows. Thus, with the added second term in (22), the limitations of the Hopf formalism have been totally removed. However small F may be, it cannot be neglected, since the second-order differential term characterizes the mathematical quality of the equation. This fact may be explained as follows.

Let us express (22) in the same way as (10),

$$\frac{\partial p^M}{\partial t} = -\sum_{i=1}^{M} \frac{\partial}{\partial b_{\alpha i}} (\chi^M_{\alpha i} p^M) + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{\partial^2}{\partial b_{\alpha i}\, \partial b_{\beta j}} (F^M_{\alpha\beta ij} p^M) \tag{23}$$

Assume that by means of the proper orthogonal transformation of the space $R^{3M} \ni \{b_{\alpha i}\}$ (note the symmetry of F), we have the diagonal form of (23) as

$$\frac{\partial p^M}{\partial t} = -\sum_{i=1}^{M} \frac{\partial}{\partial \tilde{b}_{\alpha i}} (\tilde{\chi}_{\alpha i}^M p^M) + \frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial \tilde{b}_{\alpha i}^2} (\tilde{F}_{\alpha i}^M p^M) \tag{24}$$

Then, each realization of $u^M(x)$ in this Markovian stochastic process can be constituted by the following generalized random walk $\{\Delta b_{\alpha i}\}$ in time Δt in the space R^{3M}.

$$\Delta \tilde{b}_{\alpha i} = \tilde{\chi}_{\alpha i}^M \, \Delta t + (\tilde{F}_{\alpha i}^M \, \Delta t)^{1/2} \imath \tag{25}$$

where $\imath$ is a standard normal random number. This relation may be derived by constructing the formal solution of p^M using the propagation kernel method.(11),[2] Here, it is to be noted that the second term is proportional to $\Delta t^{1/2}$. Thus, however small $\tilde{F}^M$ may be, this term always dominates the first (dynamical) term as $\Delta t \to 0$. As a result, we can no longer consider $u(x) = \sum b_i s_i(x)$ as differentiable with respect to time in the strict sense! Only in the average sense may we consider the time differentiability of u, when χ_α is a sufficiently stable operator against a small randomization by the second term.

The explicit form of F representing the effect of thermal agitation will be given in the next section. Besides the natural thermal agitation, we may consider a special physical device to generate a different random force acting for fluids when different types of F, G, etc., are given. Otherwise, apart from the physics, an interesting possibility of generating a random force unconsciously is worthwhile pointing out. It comes in the form of roundoff or truncation error in the process of computer work. Suppose that we compute an ensemble of solutions of the Navier–Stokes equation (without any random force), point by point of time. Then, such an error plays the role of *some kind of* random force term for the development of $u(x)$ at every point of time. As a result, the (sufficiently large) ensemble of calculated solutions would look time-irreversible, while the theoretical basis has been understood to be completely within the Hopf formalism. However, the explicit form of Γ in this case seems to be unexplored, but is worth studying.

Generally, it is obvious that the final steady state of our ensemble depends on χ as well as F. However, if F is extremely small comparing to $\chi(u)$, the effect of this is considered to be limited to such a small region of the space of u that p has a sharp change of value where the second term on the right-hand side of (22) dominates. This offers a typical problem of a singular perturbation. In this case, the global quality of the stationary ensemble would be rather insensitive to F; F would act just as a *continuous* trigger to cause a turbulence for a supercritical Reynolds number, while it would remain as a trivial random noise around a laminar flow for a subcritical Reynolds number. As one example, we present some results of a calculation

[2] See Ref. 19 for an example of such a calculation.

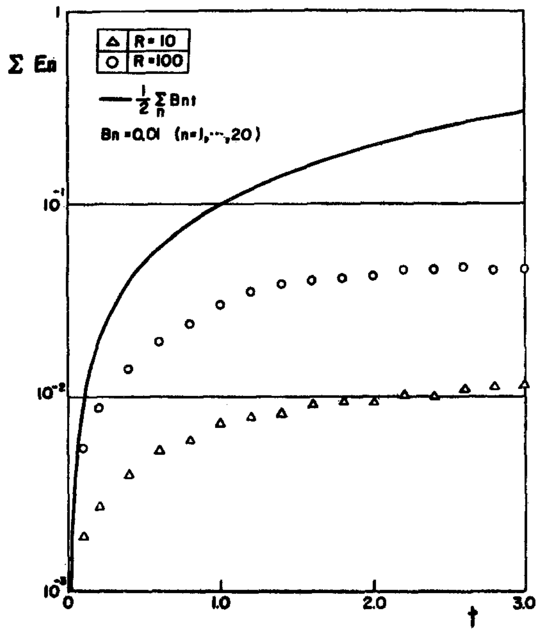

Fig. 1. Time development of the energy of the stochastic Burgers (secondary) flow with white noise.

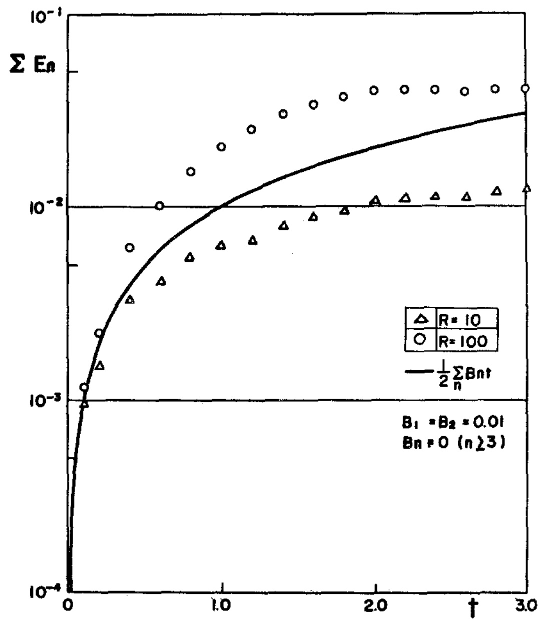

Fig. 2. Time development of the energy of the stochastic Burgers (secondary) flow with red noise.

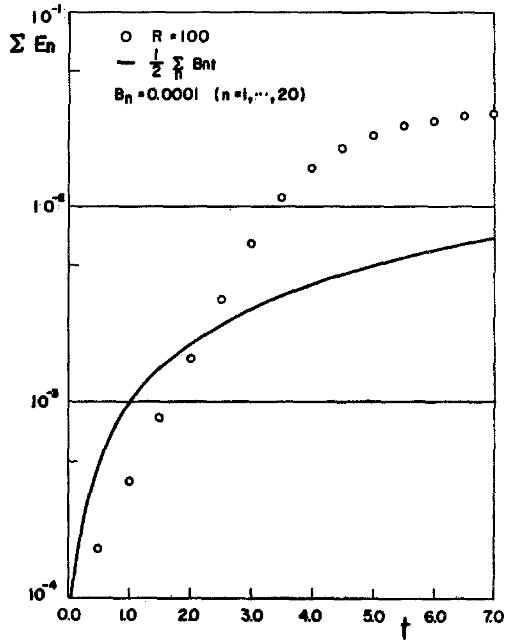

Fig. 3. Time development of the energy of the stochastic Burgers (secondary) flow with white but very weak noise.

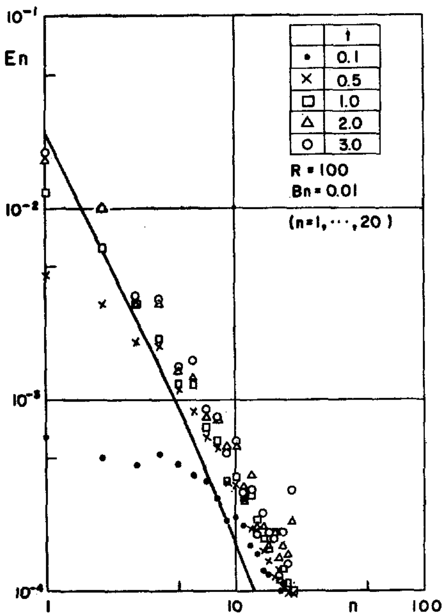

Fig. 4. Energy spectrum of the stochastic Burgers (secondary) flow with white noise for Re = 100.

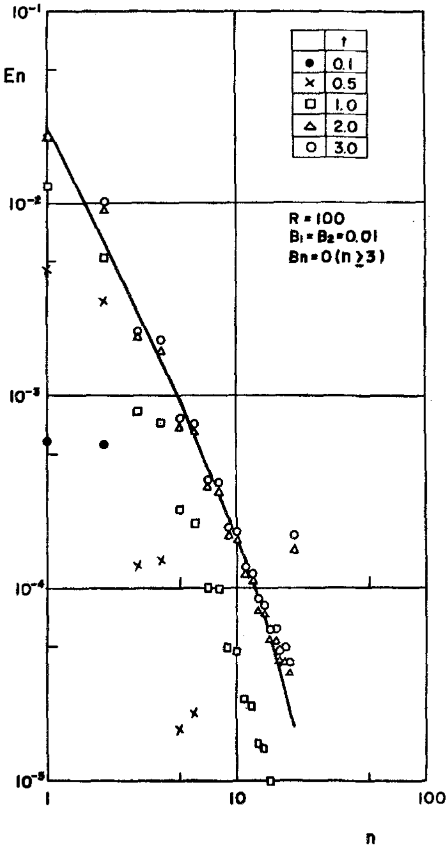

Fig. 5. Energy spectrum of the stochastic Burgers (secondary) flow with red noise for Re = 100.

for the random-forced, one-dimensional, space-limited, main-flow interacting Burgers flow in Figs. 1–6. In this case the Burgers operator

$$\chi(u) = u - 2u\frac{\partial u}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial u^2}{\partial x^2} \tag{26}$$

replaces the Navier–Stokes operator, where $x \in [0, 1]$. F is given as

$$F(x, x') = \sum_{n=1}^{20} B_n \sin n\pi x \sin n\pi x' \tag{27}$$

(Note that the space-limited Burgers flow is not homogeneous in space.) For supercritical Reynolds number, Re = 100, the velocity development of an ensemble of 200 realizations was calculated, using (25), starting from the laminar state $u(x) = 0$ (with modes cut off beyond the 20th), for the cases of (a) all $B_n = 0.01$, (b) $B_1 = B_2 = 0.01$ but other B_n vanishing, and (c) all $B_n = 0.0001$. The details were recently published.[12] Figures 1–3 show the development of total turbulence energy for cases (a)–(c), according to which

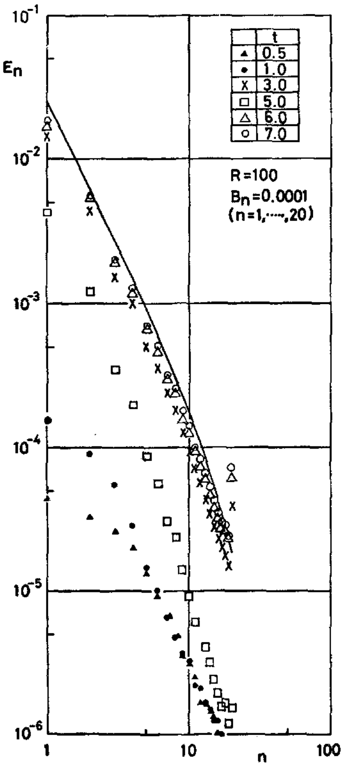

Fig. 6. Energy spectrum of the stochastic Burgers (secondary) flow with white but very weak noise for Re = 100.

the energy levels of the steady state of the three turbulences are rather the same in spite of a big difference in the input random-force power and in the type of F. In case (b), the turbulence energy grows rapidly beyond the input random-force energy. Figures 4–6 show the energy spectra of the three cases. (The solid line in the figures indicates, for reference, one steady solution of the Burgers equation which has nearly the k^{-2} spectrum.) At the steady states, the same overall characteristics are seen in all cases. These results support the present view of the effect of F. Thus, we may expect a random-force independence law (as $F \to 0$ but $\neq 0$) to appear for a lower moment feature of the distribution p, i.e., for lower order correlations of fluid variables, even for the case of the random-forced Navier–Stokes flow. [This reminds us of the Reynolds-number independence law (as Re $\to \infty$), which was found for the energy spectrum and decay feature of the space-unlimited, one-dimensional Burgers flow.[20]]

4. FORM OF NATURAL RANDOM FORCE

There are noteworthy discussions by Landau and Lifshitz,[1] Fox and Uhlenbeck,[2] Kelly and Lewis,[3] and Betchov[4] of what random force should be added to the Navier–Stokes equation. They concluded that the random force is stationary, homogeneous, and Gaussian. The first three works coincide in deriving the correlation

$$F_{\alpha\beta}(x - x') = 2\frac{\kappa T}{\rho}\left[\nu\left(\delta_{\alpha\beta}\frac{\partial^2}{\partial x_\gamma\,\partial x_\gamma'} + \frac{\partial^2}{\partial x_\beta\,\partial x_\alpha'}\right) + \left(\zeta - \frac{2}{3}\nu\right)\frac{\partial^2}{\partial x_\alpha\,\partial x_\beta'}\right]\delta(x - x') \tag{28}$$

where κ is the Boltzmann constant, T is the absolute temperature, ρ is the density of the fluid, ν is the usual kinematic viscosity, and ζ is the second kinematic viscosity. Furthermore, they derived the correlation of the random force to the heat flux vector. Since the present paper concerns only incompressible flows, this item will be omitted here. [To include compressible flows, we have only to take into account the field dynamics of density and other thermodynamic variables as $\chi_4, \chi_5, \ldots$, and then (22) is still useful, if the full valid form of F is given according to the works of these authors.]

Although (28) is sufficiently supported by proper physical studies, it is a surprise to fluid dynamicists studying incompressible flows that (28) shows no solenoidality of the random force field. This means that there exist no incompressible flows in the strict sense once the effect of molecular fluctuation, which is called thermal agitation, is taken into account in the dynamics! However, in order to avoid the mathematical complication of having to treat another field dynamics of the density, we prefer to keep the concept of solenoidality in this paper by deforming (28) in a solenoidal way. That is,

$$F_{\alpha\beta}(x - x') = 2\frac{\kappa T\nu}{\rho}\left(\delta_{\alpha\beta}\frac{\partial^2}{\partial x_\gamma\,\partial x_\gamma'} - \frac{\partial^2}{\partial x_\beta\,\partial x_\alpha'}\right)\delta(x - x') \tag{29}$$

Exactly corresponding to this, we have the correlation of wave components of the random force field as follows:

$$\langle g_\alpha(k, t) g_\beta^*(k', t)\rangle \equiv \delta(t - t')F_{\alpha\beta}(k, k') \tag{30}$$

with

$$F_{\alpha\beta}(k, k') = \frac{1}{(2\pi)^3}\frac{2\kappa T\nu}{\rho}\,\delta(k - k')(\delta_{\alpha\beta}k^2 - k_\alpha k_\beta) \tag{31}$$

($k^2 = k_\gamma k_\gamma$), where we have put

$$f(x, t) = \int g(k, t)\exp(ik_\alpha x_\alpha)\,dk \tag{32}$$

and note $g^*(k, t) = g(-k, t)$ for the reality condition. It is worthwhile to note that (31) is in accord with Betchov's formula except that his numerical factor is 3 rather than 2.

The present functional formalism can help us also in searching for a most plausible form of F, as follows. In terms of the argument function in wave-number space, (19) in the present case may be written as

$$\frac{\partial\psi}{\partial t} = i\int z_\alpha^*(k)\chi_\alpha\left[\frac{\delta}{i\,\delta z^*(k)}\right]\psi\,dk - \frac{1}{2}\iint z_\alpha^*(k)z_\beta(k')F_{\alpha\beta}(k,k')\psi\,dk\,dk' \tag{33}$$

where

$$\chi_\alpha(v) = -ik_\gamma\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\int v_\beta(k-k')v_\gamma(k')\,dk' - \nu k^2 v_\alpha(k) \tag{34}$$

[In order to derive (33) directly from (19), put $y(x) = [1/2(\pi)^3\int z(k)\exp(ik_\alpha x_\alpha)\,dx$ and $u(x) = \int v(k)\exp(ik_\alpha x_\alpha)\,dk$, noting the reality condition on z and v.] Considering the case where the velocity is so small as to be comparable in magnitude with the random force, and where the nonlinear term in (34) is completely negligible, we have the simple functional equation for the steady state

$$-\nu k^2\frac{\delta\psi_\infty}{\delta z_\alpha^*(k)} - \frac{1}{2}z_\beta(k)D_{\alpha\beta}(k)\psi_\infty = 0 \tag{35}$$

assuming the wave number independence of random forces, $F_{\alpha\beta}(k,k') = \delta(k-k')D_{\alpha\beta}(k)$. The solution of (35) is easily found as

$$\psi_\infty = \exp\left[-\tfrac{1}{2}\int z_\alpha^*(k)z_\beta(k)D_{\alpha\beta}(k)/(2\nu k^2)\,dk\right] \tag{36}$$

keeping $D_{\beta\alpha}(-k) = D_{\alpha\beta}(k)$ in mind, which follows from the general definition (30). This yields

$$\langle v_\alpha(k)v_\beta^*(k')\rangle = \left.\frac{\delta^2\psi}{i^2\,\delta z_\alpha^*(k)\,\delta z_\beta(k')}\right|_{z=0} = \frac{\delta(k-k')D_{\alpha\beta}(k)}{2\nu k^2} \tag{37}$$

This is nothing but the fluctuation-dissipation theorem in our fluid motion. If the velocity is solenoidal, $D_{\alpha\beta}$ is required to be of the form

$$D_{\alpha\beta}(k) = B(k)\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) \tag{38}$$

As a result, we have the energy spectrum of fluid motion per unit mass as

$$E(k) = \tfrac{1}{2}B(k)/\nu k^2 \tag{39}$$

Here, if we employ a consideration similar to Einstein's for the Brownian motion that energy is equipartitioned to each degree of freedom of collective fluid motion at thermal equilibrium, we can put

$$\rho L^3 E(k)\,\Delta k = \kappa T \tag{40}$$

Here, the box approach (of length L) to physical space was retained to discretize wavenumber space. Note that the right-hand side is different from $\frac{3}{2}\kappa T$, because $v(k)$ has only two degrees of freedom on account of solenoidality. Keeping in mind that $\lim_{L\to\infty} L^3 \, \Delta k = (2\pi)^3$ and combining the relations (38)–(40), it is clear that we have reproduced the expression (31). It is obvious that Betchov made an error of overcounting the degrees of freedom, forgetting the solenoidality. (Note that the total degrees of freedom of wave motion must be far less than those of all kinetic molecular motions, so that we have a reason to consider a certain cutoff wave number. This suggests that the noncylinder limit of ψ described in the previous section is only the ideal limit based on the continuum concept, but is not practically reached.)

Thus, the forms of F in (29) and (31) are most universal as representing the natural random force in our incompressible flow to the best of our knowledge, even though they have a slightly unrealistic aspect compared to (28). If the flow is far from local thermal equilibrium, the present form may break down as Betchov pointed out; but in such a highly nonequilibrium case, the Navier–Stokes equation itself loses its validity, so that it is apparent that every formulation should start again from the more fundamental basis of statistical physics. Finally, it may be noted that our ensemble mechanics with the natural random force action reduces simply to the Hopf equation for the inviscid flow case with $\nu = 0$, where there is *no* irreversibility. This is important in showing that viscosity takes charge not only of the simple irreversibility of the ensemble behavior, but also in ensuring the whole mechanics to be such a Markovian stochastic process as governed by a Fokker–Planck equation [as (22)], which guarantees achievement of a unique ultimate steady state of the ensemble.

5. CONCLUDING REMARKS

It is unfortunate that no standard analytical technique for solving a functional-differential equation exists at present. All particular solutions of the ensemble mechanics are left as tasks for the future. We may have a simple approach, like a kinetic equation approach to the Liouville dynamics, which deals only with lower order correlations of fluid variables. But it is desirable that such an approach be substantially correlated with the ensemble mechanics in taking valid account of the stochastic aspect of the process. On the other hand, a direct approach to the ensemble mechanics by numerical simulation [such as suggested by the formula (25)] is also possible. The success of this approach seems to depend on an efficient algorithm executable on a present-day computer. Finally, it is stressed that this paper does not deny the significance and value of the conventional Navier–Stokes equation in laminar flow analysis, but indicates a direction for improving the Hopf formalism in order to involve turbulent flow phenomena in a rational way.

APPENDIX—ENTROPY OF ENSEMBLE MECHANICS

The probability equation corresponding to (33) is written in the same way as (22),

$$\frac{\partial p}{\partial t} = -\int \frac{\delta}{\delta v_\alpha(k)} [\chi_\alpha(v)p]\, dk + \tfrac{1}{2}\iint \frac{\delta^2}{\delta v_\alpha(k)\, \delta v_\beta{}^*(k')} [F_{\alpha\beta}(k, k')p]\, dk\, dk' \quad \text{(A.1)}$$

According to Shannon, the H function of our information on the ensemble may be introduced as

$$H(p) = \lim_{M\to\infty} \int p^M \log p^M \, \delta v^M/M \quad \text{(A.2)}$$

where δv^M is defined in the same way as δu^M [cf. (9)] but taking into account the cross-product elements of the real and imaginary parts of v^M. Omitting M in (A.2) and using a simple functional calculus, we proceed with the calculation of dH/dt:

$$\begin{aligned}\frac{dH}{dt} &= \int \frac{\partial p}{\partial t} \log p \, \delta v \\ &= \int \left\{ \int \chi_\alpha(v) \frac{\delta p}{\delta v_\alpha(k)}\, dk \right.\\ &\quad \left. - \frac{1}{2}\iint \left[F_{\alpha\beta}(k, k') \frac{\delta p}{\delta v_\alpha(k)} \frac{\delta p}{\delta v_\beta{}^*(k')} p^{-1} \right] dk\, dk' \right\} \delta v \end{aligned} \quad \text{(A.3)}$$

unless $F_{\alpha\beta}$ depends on v. Hence, if the expression (34) is inserted, we have the first term in (A.3) equal to

$$\iint \left\{ \left[2ik_\gamma v_\gamma(0) + i\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) k_\alpha v_\beta(0) \right] p - \nu k^2 v_\alpha(k) \frac{\delta p}{\delta v_\alpha(k)} \right\} dk\, \delta v \quad \text{(A.4)}$$

the first (square-bracketed) term of which vanishes (note that $k_\gamma v_\gamma$ is odd in k). As a result,

$$\begin{aligned}\frac{dH}{dt} &= \int \left\{ \int \left[-\nu k^2 v_\alpha(k) \frac{\delta p}{\delta v_\alpha(k)} \right] dk \right.\\ &\quad \left. - \frac{1}{2}\iint \left[F_{\alpha\beta}(k, k') \frac{\delta p}{\delta v_\alpha(k)} \frac{\delta p}{\delta v_\beta{}^*(k')} p^{-1} \right] dk\, dk' \right\} \delta v \end{aligned} \quad \text{(A.5)}$$

Thus, without any viscosity and random force effects, the entropy as the negative H function should be invariant with respect to time. A further calculation yields

$$\text{first square-bracketed term in (A.5)} = \int \nu k^2 \delta(0)\, dk \qquad (A.6)$$

in which $\delta(0)$ is understood as $1/\Delta k$ in the limit $\Delta k \to 0$. Although the term is indefinite in the *continuum* limit, it is obvious that viscosity makes the entropy decrease without limit and so steady entropy cannot be reached. These are the results of Tatsumi and Ikeda.[13]

However, if there is a random force, the result is different. Since F is positive definite, it is clear that the second term in (A.5) is negative. Therefore, the entropy decrease should finally be stopped at the steady state, and if the initial entropy is too small, the entropy should increase until H is saturated. This is assured by constructing the p_∞ corresponding to ψ_∞ in (36). Corresponding to (35), we have

$$\nu k^2 v_\alpha(k) p_\infty + \frac{1}{2} D_{\alpha\beta}(k) \frac{\delta p_\infty}{\delta v_\beta{}^*(k)} = 0 \qquad (A.7)$$

which gives

$$p_\infty = A_\infty \exp\left[- \int v_\alpha(k) v_\beta{}^*(k) \nu k^2 / D_{\alpha\beta}(k)\, dk \right] \qquad (A.8)$$

where A_∞ is the normalization constant. Thus, we obtain

$$\frac{dH_\infty}{dt} = \iint \frac{\delta p_\infty}{\delta v_\alpha(k)} \left[-\nu k^2 v_\alpha(k) - \frac{1}{2} D_{\alpha\beta}(k) \frac{\delta p_\infty}{\delta v_\beta{}^*(k)} p_\infty^{-1} \right] dk\, \delta v = 0 \qquad (A.9)$$

because of (A.7). The steady value H_∞ is given by (A.2) and (A.8), and must never be a minimum. This result is in remarkable contrast to the H-theorem in statistical mechanics.

ACKNOWLEDGMENTS

The author is grateful to Dr. D. A. Lee, Aerospace Research Laboratories, Wright-Patterson AFB, for his fruitful discussions and courteous hospitality. The work was done while the author held an NRC–AFSC Senior Research Associateship.

REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, New York (1959), Chap. 17.
2. R. F. Fox and G. E. Uhlenbeck, *Phys. Fluids* **13**:1893 (1970).
3. G. E. Kelly and M. B. Lewis, *Phys. Fluids* **14**:1925 (1971).

4. R. Betchov, in *Rarefied Gas Dynamics*, L. Talbot, ed., Academic Press, New York (1961), p. 307.
5. R. H. Kraichnan, *Phys. Rev.* **109**:1407 (1958); *J. Fluid Mech.* **5**:497 (1959).
6. E. A. Novikov, *Soviet Physics—JETP* **20**:1290 (1965).
7. S. F. Edwards, *J. Fluid Mech.* **18**:239 (1968).
8. I. Hosokawa, *J. Phys. Soc. Japan* **25**:271 (1968).
9. G. K. Batchelor, *Theory of Homogeneous Turbulence*, Cambridge University Press, New York (1953).
10. E. Hopf, *J. Rat. Mech. Anal.* **1**:87 (1952).
11. I. Hosokawa, *J. Math. Phys.* **8**:221 (1967).
12. I. Hosokawa and K. Yamamoto, *J. Stat. Phys.* **13**:245 (1975).
13. T. Tatsumi and N. Ikeda, Inst. Space–Aero. Sci., Univ. Tokyo, Rep. 2, A-73 (1966).
14. I. Hosokawa, *Prog. Theor. Phys.* **53**:1517 (1975).
15. E. Hopf and E. W. Titt, *J. Rat. Mech. Anal.* **2**:587 (1953).
16. R. H. Kraichnan, *J. Math. Phys.* **2**:124 (1961); J. R. Herring and R. H. Kraichnan, in *Statistical Models and Turbulence*, M. Rosenblatt and C. Van Atta, eds., Springer-Verlag, New York (1971), p. 148.
17. U. Frisch, M. Lesieur, and A. Brissaud, *J. Fluid Mech.* **65**:145 (1974).
18. A. H. Gray, Jr., *J. Math. Phys.* **6**:644 (1965).
19. I. Hosokawa, *J. Math. Phys.* **11**:657 (1970).
20. I. Hosokawa and K. Yamamoto, *Phys. Fluids* **13**:1683 (1970).